Solution to an Unsolved Sangaku Geometric Puzzle: Morikawa's Problem

John Andraos

E-mail: johnandraos1964@gmail.com

Abstract. A solution is presented for an unsolved sangaku problem proposed by Morikawa Jihei at a Tenman shrine.

Keywords. Morikawa sangaku problem, Japanese mathematics

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1. INTRODUCTION

Sangaku problems are fascinating Japanese temple problems based on geometric constructions involving tangent circles, inscribed triangles and circles, and the like. Such problems were written on wooden tablets which were placed as offerings and challenges to others at Shinto shrines and Buddhist temples during the Edo period in Japan (1603-1868). For readers who are unfamiliar with the subject, a rich literature exists [1-19] that has attracted the attention of mathematicians, teachers of mathematics, historians of mathematics, and others from various disciplines who have an appreciation of the beauty of geometry. In this work we propose a solution to one unsolved sangaku problem posed by Morikawa Jihei at a Tenman shrine. It is not known when this problem was posed and there is no available biographical information on Morikawa Jihei to set the context of what mathematical techniques were known to him that he or his contemporary problem solvers could use to solve the problem. What is known about these Wasan problems is that the techniques used to solve them involved geometric constructions and algebraic manipulations, but did not involve modern techniques of computational numerical analysis.

2. STATEMENT OF MORIKAWA PROBLEM

Two circles with radii a and b sit on the line L and touch each other. Between them is an inscribed square with side x as shown in Figure 1. Find the minimum of x in terms of a and b. This problem appears as problem #18 in Chapter 7 of Fukagawa and Rothman's book [8].

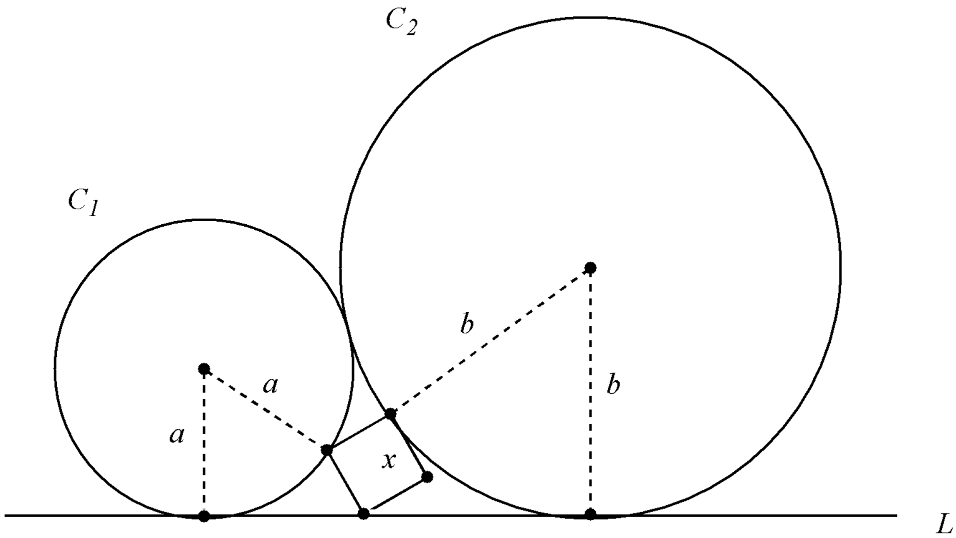

FIGURE 1. Diagram pertaining to Tenman shrine sangaku proposed by Morikawa Jihei.

There are two peculiarities about the stated problem. Firstly, the word "minimum" is used to describe the size of the sought after inscribed square suggesting that there exist several possible squares that can be fit as shown in the enclosed space where only one of them is a minimum. One would think at first that fitting a unique *maximum*-sized square in the space enclosed by the two circles and line L would be appropriate. Secondly, the problem does not impose any stated constraint on the orientation of the tilted square. We notice however that the upper two corners of the square each touch the smaller and larger circles, respectively and the third corner lies on line L. The problem suggests that it could be considerably simplified by rotating the inscribed square so that one side lies on line L as shown in Figure 2. In this case it is not hard to see that fitting the maximum-sized square in the enclosed space is indeed appropriate. We do not know, however, if Morikawa had thought or had intended that this rotation manipulation would lead to the same sized square for both cases. In our approach we first investigate the solution of finding the maximum value of *x* of a horizontal oriented square placed between the two circles in terms of the radii *a* and *b*. Learning from that solution we then investigate the original Morikawa problem of a tilted square with the specified constraint that the two upper corners of the square must lie on the respective circles and that the third corner must lie on line L.

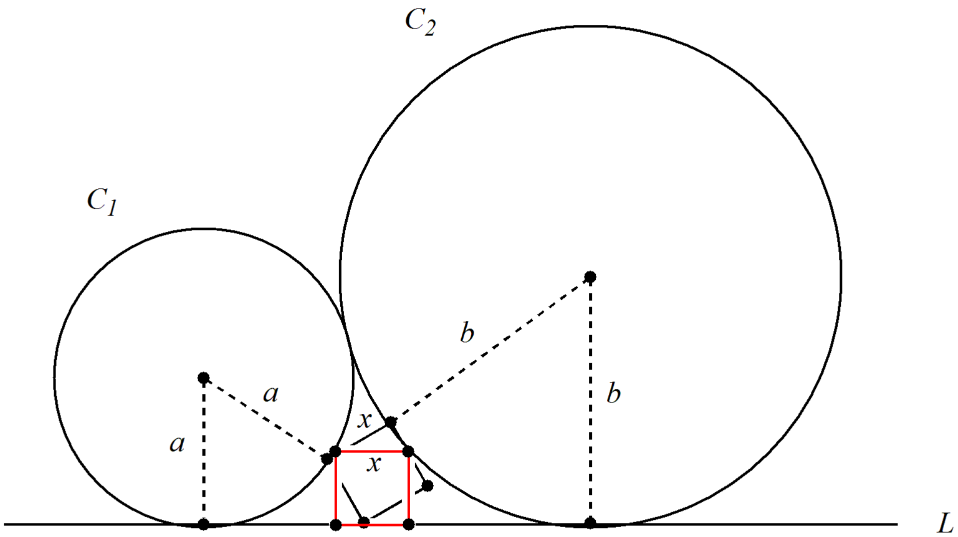

FIGURE 2. Diagram pertaining to Tenman shrine sangaku showing rotated inscribed square.

3. SOLUTION TO FINDING THE MAXIMUM-SIZED HORIZONTAL SQUARE

Figure 3 shows the diagram pertaining to the simplified problem.

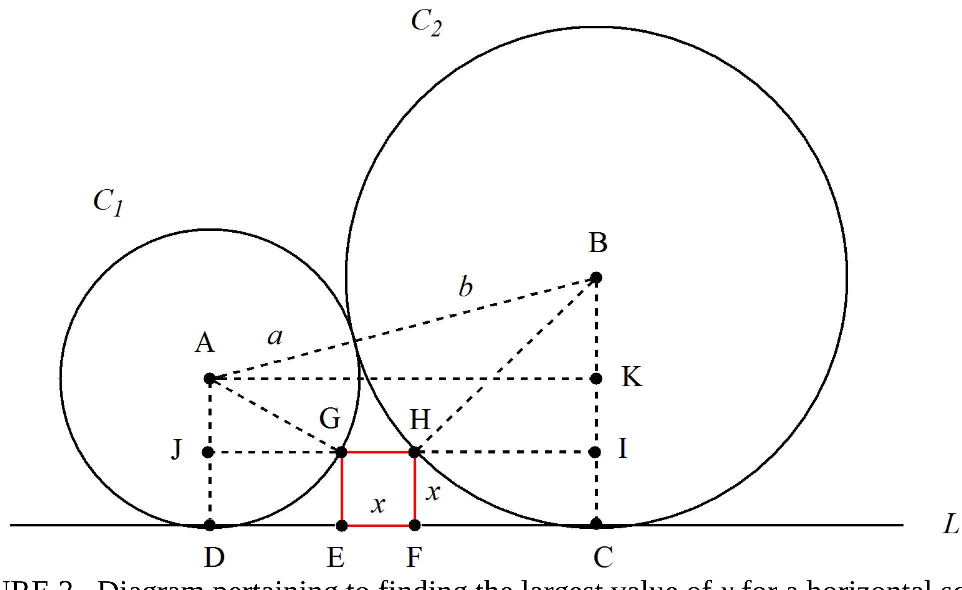

FIGURE 3. Diagram pertaining to finding the largest value of x for a horizontal square.

Based on Figure 3 we have the following lengths:

$\overline{AJ} = a - x \qquad \overline{AG} = a$

$\overline{BI} = b - x \qquad \overline{BH} = b$

Applying the Pythagorean theorem to triangles *AJG* and *BIH*, we have the following lengths:

$$\overline{JG} = \sqrt{a^2 - (a-x)^2} = \sqrt{2ax - x^2}$$

$$\overline{HI} = \sqrt{b^2 - (b-x)^2} = \sqrt{2bx - x^2}$$

However, we also see that $\overline{JG} = \overline{DE}$ and $\overline{HI} = \overline{FC}$.

In triangle *ABK* we have $\overline{AB} = a + b$ and $\overline{BK} = b - a$. Applying the Pythagorean theorem again to this triangle leads to

$$\overline{AK} = \sqrt{(a+b)^2 - (b-a)^2} = 2\sqrt{ab}$$

Since $\overline{DC} = \overline{AK}$, we can write the relation given by equation (1)

(1) $$2\sqrt{ab} = x + \sqrt{2ax - x^2} + \sqrt{2bx - x^2}$$

which can be transformed to the quartic polynomial given by equation (2)

(2) $$x^4 + Ax^3 + Bx^2 + Cx + D = 0$$

where the constants are given by equations (3a) to (3d).

(3a) $$A = -\frac{4}{5}\left(a + b + 6\sqrt{ab}\right)$$ (3b) $$B = \frac{4}{5}\left(8ab + a^2 + b^2 + 4\sqrt{ab}\,(a+b)\right)$$

(3c) $$C = -\frac{16ab}{5}\left(2\sqrt{ab} + (a+b)\right)$$ (3d) $$D = \frac{16a^2b^2}{5}$$

If we let $\frac{b}{a} = s$, then the above constants are given by equations (4a) to (4d).

(4a) $$A = -\frac{4}{5}a\left(1 + s + 6\sqrt{s}\right)$$ (4b) $$B = \frac{4}{5}a^2\left(8s + 1 + s^2 + 4(1+s)\sqrt{s}\right)$$

(4c) $$C = -\frac{16}{5}a^3 s\left(2\sqrt{s} + 1 + s\right)$$ (4d) $$D = \frac{16}{5}a^4 s^2.$$

Case I

We can explore the scenario when the two circles have the same size, that is, when $b = a$ or $s = 1$. In this case, the quartic polynomial reduces to equation (5)

$$(5) \qquad x^4 - \left(\frac{32}{5}a\right)x^3 + \left(\frac{72}{5}a^2\right)x^2 - \left(\frac{64}{5}a^3\right)x + \frac{16}{5}a^4 = 0$$

whose real roots are $x = \frac{2}{5}a$ and $x = 2a$. Since from Figure 3 we see that $x < a$ and therefore the sensible root is $x = \frac{2}{5}a$. This result can also be verified from the diagram shown in Figure 4.

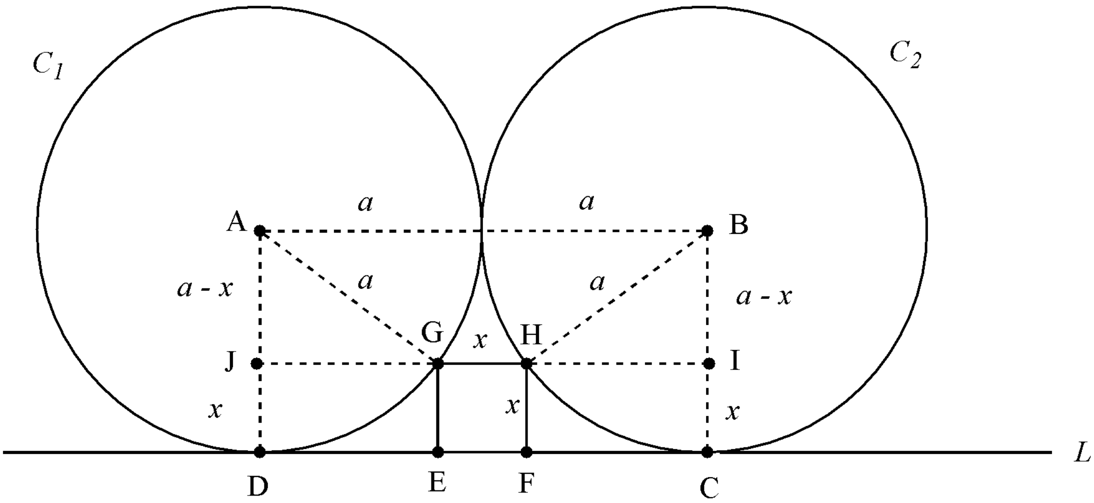

FIGURE 4. Fitting the largest inscribed square between equal-sized tangent circles and line L.

Following the same sequence of steps as before we find that $\overline{AB} = \overline{DC} = 2a$ and

$2a = x + 2\sqrt{2ax - x^2}$ which can be transformed to the quadratic equation given by

$5x^2 - 12ax + 4a^2 = 0$. This can be factored to $(5x - 2a)(x - 2a) = 0$ from which we find the two real roots $x = \frac{2}{5}a$ and $x = 2a$. As before, since from the Figure 4 $x < a$, then the root that is sensible is $x = \frac{2}{5}a$ which agrees with the result found from the general quartic equation (5).

<u>Case II</u>

As an aside, we next investigate the case when the tangent point of the two circles (point O) is exactly collinear with points G and E in Figure 3 as shown in Figure 5.

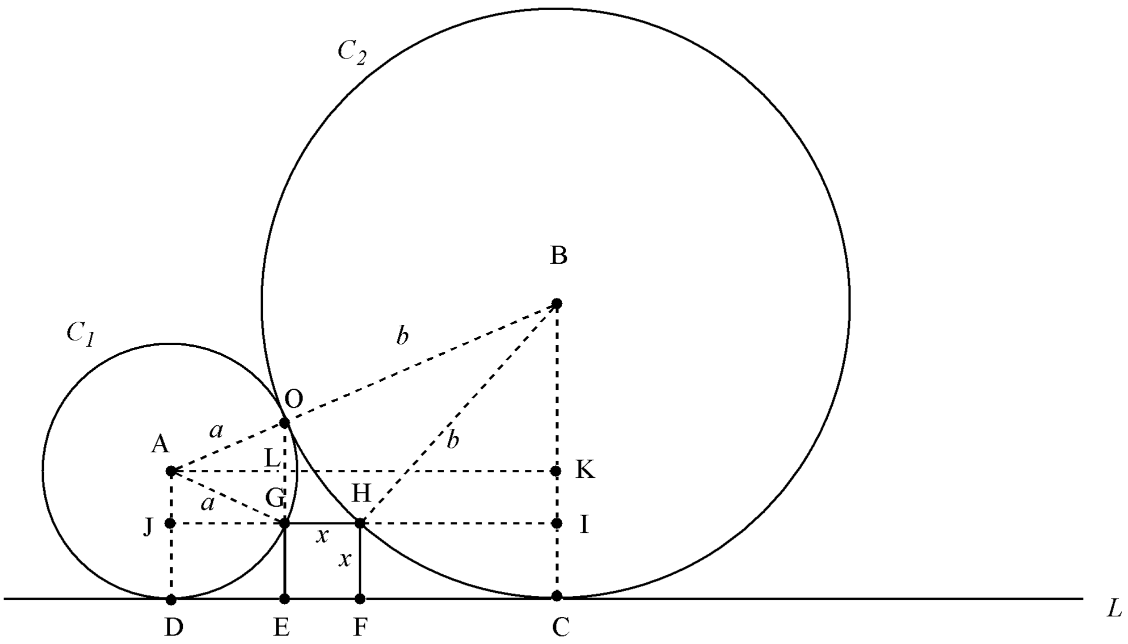

FIGURE 5. Tangent circles with an inscribed square whose upper left corner is in line with the point of tangency of the two circles.

From Figure 5 we note the following lengths: $\overline{JD} = x, \overline{AD} = \overline{AO} = \overline{AG} = a, \overline{AJ} = \overline{LG} = a - x, \overline{AL} = \overline{JG} = \overline{DE}$. Applying the Pythagorean theorem in succession to triangles *ALG*, *AOL*, and *OGH* yields $\left(\overline{AL}\right)^2 = 2ax - x^2$, $\overline{OL} = a - x$, and $\left(\overline{OH}\right)^2 = 4a^2 - 8ax + 5x^2$. If we set $\overline{OH} = a$, then we obtain a quadratic equation given by $5x^2 - 8ax + 3a^2 = (5x - 3a)(x - a) = 0$ whose roots are $x = \frac{3}{5}a$ and $x = a$. Since $x < a$, the viable root is $x = \frac{3}{5}a$. From this result we can determine the value of *s* and hence the relative size of the radius of circle C_2 using the quartic equation (2) and equations (4a) to (4d). Hence, when we substitute $x = \frac{3}{5}a$ into equation (2) we obtain

(6)

$$\frac{3^4}{5^4}a^4 - \left(\frac{2^2 3^3}{5^4}\left(1 + s + 6\sqrt{s}\right)\right)a^4 + \left(\frac{2^2 3^2}{5^3}\left(1 + 8s + s^2 + 4(1+s)\sqrt{s}\right)\right)a^4 - \left(\frac{2^4 3}{5^2}s\left(1 + s + 2\sqrt{s}\right)\right)a^4 + \frac{2^4}{5}a^4 s^2 = 0$$

which simplifies after some algebra to equation (7).

(7) $$960400s^4 - 2563680s^3 + 559224s^2 + 35208s + 23409 = 0$$

The software program GraphCalc 4.0.1 produces the roots

s = 0.37274571942552 or s = 2.4208906242747.

Since circle C_2 is larger than circle C_1 the viable root is s = 2.4208906242747.

We can obtain the same result analytically from equation (1) by first solving for b yielding equation (8)

(8) $$b^2\left(\alpha^2-2\beta^2\right)+b\left(\beta^2 x-2\alpha\left(\phi^2+x^2\right)\right)+\left(\phi^2+x^2\right)^2=0$$

where $\alpha=2(2a+x)$, $\beta=4\sqrt{ax}$, and $\phi=x+\sqrt{2ax-x^2}$. After substituting $x=\frac{3}{5}a$ we obtain equation (9).

(9) $$b^2\left(\frac{196}{25}a^2\right)-b\left(a^3\frac{12\left(109+26\sqrt{21}\right)}{125}\right)+9a^4\frac{253+52\sqrt{21}}{625}=0.$$

Next, substitution of $b=sa$ and further simplification eventually leads to quadratic equation (10) in s

(10) $$s^2 4900-s60\left(109+26\sqrt{21}\right)+9\left(253+52\sqrt{21}\right)=0$$

whose real roots are $s=\frac{3\left(109+26\sqrt{21}\right)\pm12\sqrt{855+195\sqrt{21}}}{490}$. Numerical evaluation of this expression taking the positive sign leads to s = 2.4208906242747 as before.

<u>Graphical Exploration</u>:

Based on the general quartic equation (11) and varying the value of s we can construct a table of values for the real roots using the GraphCalc program. These are summarized in Table 1. The entries shown in red text correspond to cases where at least one of the x roots is exact.

(11)
$$x^4-\left(\frac{4}{5}a\left(1+s+6\sqrt{s}\right)\right)x^3+\left(\frac{4}{5}a^2\left(1+8s+s^2+4(1+s)\sqrt{s}\right)\right)x^2-\left(\frac{16}{5}a^3 s\left(1+s+2\sqrt{s}\right)\right)x+\frac{16}{5}a^4 s^2=0$$

Table 1. Summary of real roots for quartic equation (11) as a function of s.

s	x_1	x_2
0	0	0
0.0001	0.000196	0.000197

0.001	0.001854	0.001909
0.01	0.015629	0.017883
0.1	0.098884	0.162974
0.2	0.158508	0.320454
0.3	0.204726	0.480542
0.37275	**0.233328**	**0.6**
0.4	0.24323	0.645572
0.5	0.276548	0.817382
0.6	0.306073	0.9981
0.7	0.332675	1.190803
0.8	0.356937	1.400877
0.9	0.379276	1.64077
1	**0.4**	**2**
1.5	0.486138	1.687467
2	0.553096	1.634765
2.42089	**0.6**	**1.616196**
3	0.654775	1.604735
4	**0.731269**	**1.6**
5	0.792542	1.602268
6	0.843552	1.606975
7	0.887154	1.612551
8	0.925153	1.618362
9	0.958762	1.624132
10	0.988839	1.629737
20	1.183471	1.673798
50	1.416374	1.739845
100	1.562876	1.788277
1000	1.854391	1.908632
10000	1.955432	1.955432
100000	1.986442	1.986442
1000000	1.99583	1.996186

The x_1 roots are the viable roots pertaining to Figure 3.

A plot of x_1 versus s is shown in Figure 6 where the abscissa is shown in a logarithmic scale. The limiting value for x_1 is 2 when s is very large.

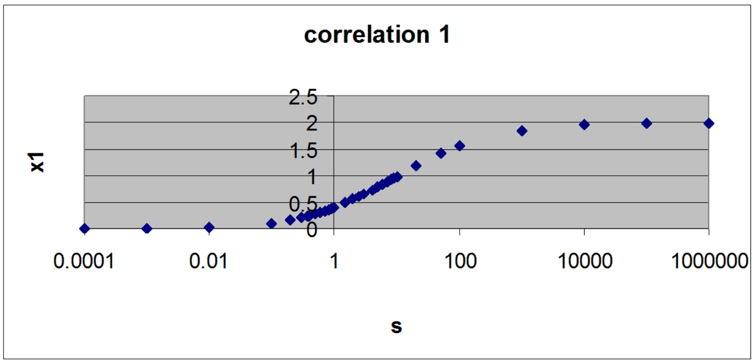

FIGURE 6. Correlation of root x_1 versus *s*.

Interestingly, a similar plot of x_2 versus *s* shows a cusp at *s* = 1 and a minimum at *s* = 4 as shown in Figure 7 where the abscissa is shown in a logarithmic scale. The minimum value of x_2 is 1.6 in the domain *s* > 1. The limiting value for x_2 is also 2 when *s* is very large.

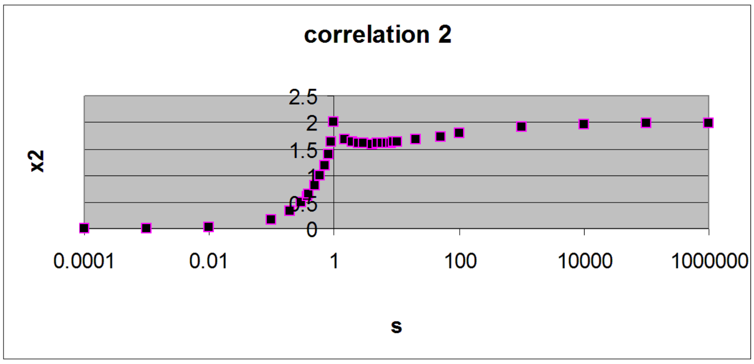

FIGURE 7. Correlation of root x_2 versus *s*.

Since the x_1 root is the meaningful solution to the problem we can fit the data given in Figure 6 to a sigmoidal function given by equation (12) which relates the dependence of the maximum value of *x* to the radii *b* and *a*.

(12) $$x1 = x_{\max} = \frac{a_1}{a_2 + 10^{-\left(a_3 z^2 + a_4 z\right)}}, \text{ where } z = \log s = \log\left(b/a\right)$$

A non-linear least squares fit using the NONLIN program [20] yields the following parameters: $a_1 = 0.4948 \pm 0.00152$, $a_2 = 0.2461 \pm 0.000908$, $a_3 = -0.02169 \pm 0.00179$, and $a_4 = 0.6188 \pm 0.00255$. A picture of the resulting data-fitting is shown in Figure 8. We note that for very large values of *s* the fitting predicts that *x*(max) approaches $a_1 / a_2 = 0.4948 / 0.2461 = 2.0106$ which is close to 2.

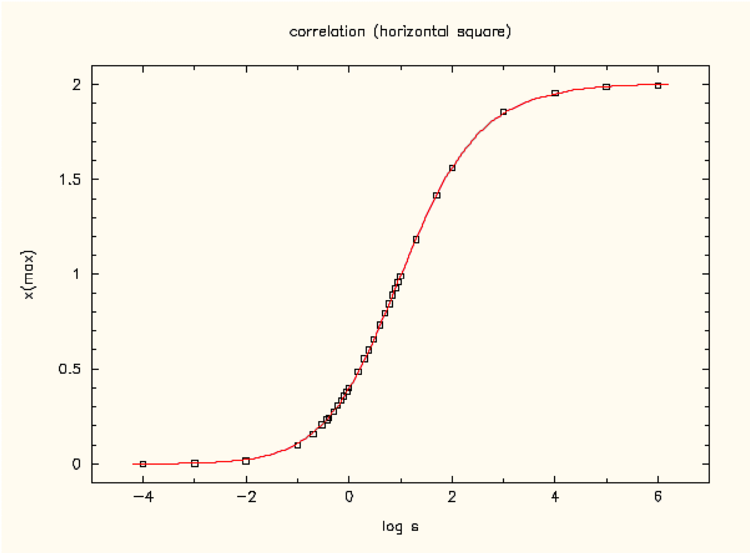

FIGURE 8. Data fit of $x_1 = x(\max)$ values versus log s according to the sigmoidal expression given in equation (12) for maximum horizontal square fit between two circles and a tangent line.

4. SOLUTION TO FINDING THE MINIMUM-SIZED TILTED SQUARE

Case I

We first investigate the case of fitting a tilted square between two equal-sized circles of radius a with the constraint that the two upper corners touch the circles and the third corner touches the bottom horizontal line. Figure 9 shows the diagram of the situation indicating the relevant lengths and angles. Lengths EC and ED are two sides of the tilted square and length CD is the diagonal.

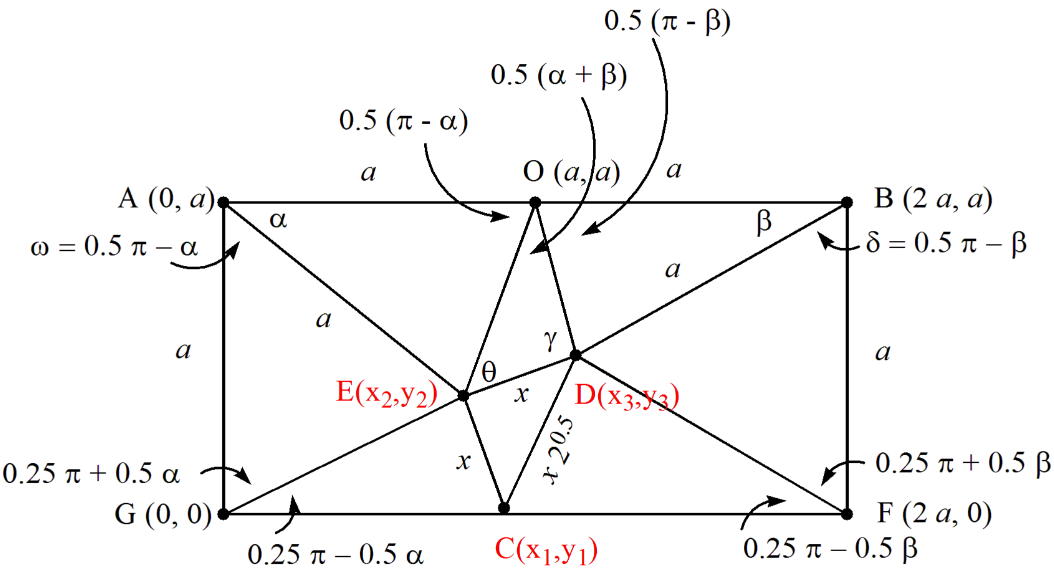

FIGURE 9. Diagram pertaining to the Morikawa problem for equal-sized circles.

The optimum value of x can be found by carrying out the following iterative cycle. For a given value of β and radius length a (set equal to 1), we start with an initial guess for x with the constraint that $x < a$.

Step 1. Obtain length CF by taking the positive root of the quadratic given by equation (13) found by applying the cosine law to triangle CDF.

(13)

$$0=\left(\overline{CF}\right)^2-2a\sqrt{1-\sin\beta}\left(\cos\left(\frac{\beta}{2}\right)+\sin\left(\frac{\beta}{2}\right)\right)\left(\overline{CF}\right)+2a^2\left(1-\sin\beta\right)-2x^2$$

Step 2. Obtain angle DCF according to equation (14) found by applying the sine law to triangle CDF.

(14)

$$\angle DCF=\sin^{-1}\left(\left(\frac{a}{x\sqrt{2}}\right)\sqrt{1-\sin\beta}\left(\cos\left(\frac{\beta}{2}\right)-\sin\left(\frac{\beta}{2}\right)\right)\right)$$

Step 3. Obtain length GC according to equation (15).

(15) $\overline{GC}=2a-\overline{CF}$

Step 4. Obtain angle ECG according to equation (16).

(16) $\angle ECG=\dfrac{3\pi}{4}-\angle DCF$

Step 5. Obtain length GE from equation (17) found by applying the cosine law to triangle GEC.

(17) $\left(\overline{GE}\right)^2=\left(\overline{GC}\right)^2+x^2-2x\left(\overline{GC}\right)\cos\angle ECG$

Step 6. Obtain angle α from equation (18) found by applying the sine law to triangle GEC.

(18) $\alpha=\sin^{-1}\left(1-\dfrac{\left(\overline{GE}\right)^2}{2a^2}\right)$

Step 7. Obtain length ED = x (new estimate of x) from equation (19) found by applying the cosine law to triangle EDO.

$$x^2 = 4a^2\left[\sin^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right] \quad (19)$$

Go back to step 1 and repeat the sequence of steps until the difference between estimates of x is less than a desired threshold. In the performed calculations this threshold was set at 1 X 10^{-8}.

Following this procedure setting a = 1, exploration of various starting values of β leads to the result that when β = 32.0414390563964°, a minimum value of x is found which is equal to 0.38528692185059. We note that this value is slightly less than x = 2/5 = 0.4 found before based on rotating the square so it lies on horizontal line GF. This means that there exists a minimum tilted square which is smaller than the maximum horizontal square that can be fit in the space between the two circles and line L. We note that such a small difference in the two x values cannot be discernable via any paper and pencil geometric construction that would have been accessible by Wasan problem solvers. From this result we can conclude that rotating the tilted square to a horizontal square does *not* lead to the same sized square that can be fit in between the two circles and the horizontal line. Moreover, the difference in size between the two kinds of squares increases as the relative sizes of the two circles increases (see Table 4 *vide infra*).

Table 2 summarizes the results of the iterative cycle for various values of β.

Table 2. Summary of x values found from iterative cycle for tilted square between two equal-sized circles.

(deg) angle β	x
5	0.707947
10	0.649353
15	0.588527
20	0.523406
25	0.450716
29	0.391181
30	0.388028
31	0.386091
31.5	0.385589
32	0.385400
32.5	0.385527
33	0.385971
34	0.387809
35	0.390906
40	0.424229
45	0.482595

50	0.558842
55	0.647164
60	0.743677
65	0.845971
70	0.952614
75	1.062826
80	1.176315

The minimum value of *x* is found by fitting an approximate quadratic function given by equation (20) to data collected between β = 29° and β = 35° and determining its minimum value using standard calculus techniques. Figure 10 shows the results of the non-linear least squares fitting using the NONLIN program [20].

(20) $$x = a_1\beta^2 + a_2\beta + a_3$$

where $a_1 = 6.27\times10^{-4} \pm 1.18\times10^{-6}$, $a_2 = -4.018\times10^{-2} \pm 7.56\times10^{-5}$, and $a_3 = 1.029 \pm 0.00121$.

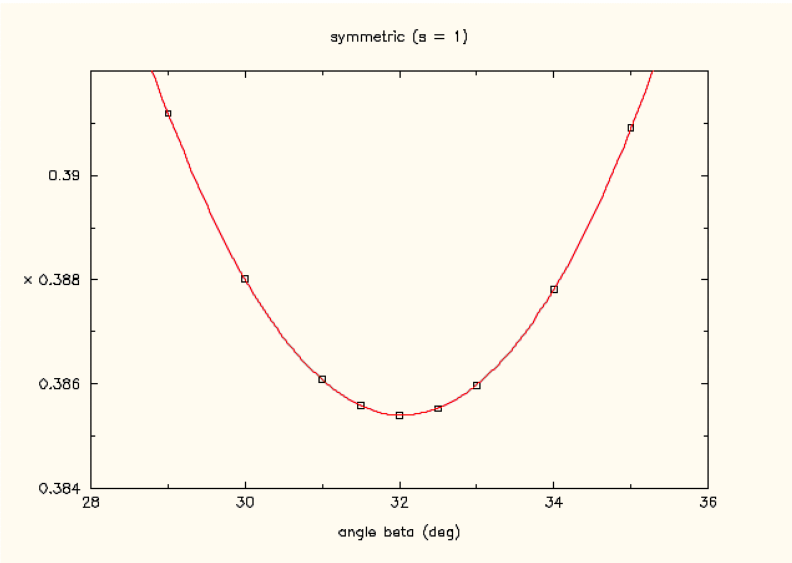

FIGURE 10. Quadratic fit according to equation (20) for data collected between β = 29° and β = 35° for a tilted square fit between two equal-sized circles.

Case II

Next, we investigate the asymmetric case of fitting a tilted square according to the same constraint as before between two unequal circles and line L. Figure 11 shows the diagram of the situation indicating the relevant lengths and angles. As before, lengths EC and ED are two sides of the tilted square and length CD is the diagonal.

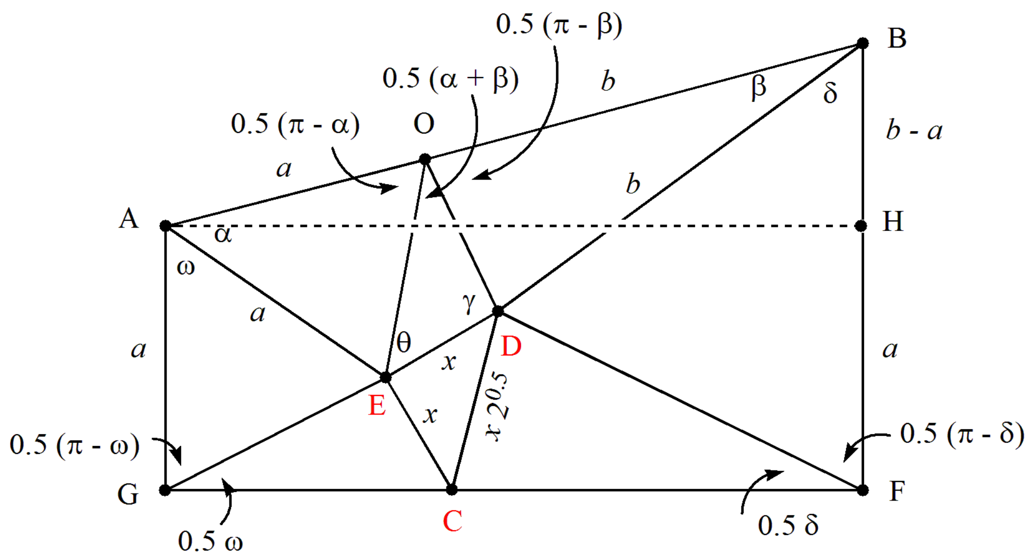

FIGURE 11. Diagram pertaining to the Morikawa problem for unequal-sized circles where $b > a$, or $s > 1$.

In this case the iterative cycle proceeds as follows. Start by defining $s = \frac{b}{a}$ with a set equal to 1 as a reference value and vary s so that $s > 1$.

Given a value of angle β we start with an initial guess for x with the constraint that $x < a$.

Step 1. Determine angle δ according to equation (21).

$$(21) \qquad \delta = \frac{\pi}{2} - \beta - \sin^{-1}\left(\frac{s-1}{s+1}\right)$$

Step 2. Determine length CF according to equation (22) found by applying the cosine law to triangle CDF.

$$(22) \qquad \overline{CF} = sa\sin\delta + \sqrt{s^2a^2\left(\sin^2\delta - 4\sin^2\left(\frac{\delta}{2}\right)\right) + 2x^2}$$

Step 3. Determine angle DCF according to equation (23) found by applying the sine law to triangle CDF.

$$(23) \qquad \angle DCF = \sin^{-1}\left[\left(\frac{sa\sqrt{2}}{x}\right)\sin^2\left(\frac{\delta}{2}\right)\right]$$

Step 4. Determine length GC according to equation (24).

$$(24) \qquad \overline{GC} = 2a\sqrt{s} - \overline{CF}$$

Step 5. Determine angle ECG according to equation (25).

$$(25) \qquad \angle ECG = \frac{3\pi}{4} - \angle DCF$$

Step 6. Determine length GE according to equation (26) found by applying the cosine law to triangle GEC.

$$(26) \qquad \left(\overline{GE}\right)^2 = \left(\overline{GC}\right)^2 + x^2 - 2x\left(\overline{GC}\right)\cos\angle ECG$$

Step 7. Determine angle ω according to equation (27).

$$(27) \qquad \omega = 2\sin^{-1}\left(\frac{\overline{GE}}{2a}\right)$$

Step 8. Determine angle α according to equation (28).

$$(28) \qquad \alpha = \frac{\pi}{2} - \omega + \sin^{-1}\left(\frac{s-1}{s+1}\right)$$

Step 9. Obtain length ED = x (new estimate of x) from equation (29) found by applying the cosine law to triangle EDO.

$$(29) \qquad x^2 = 4a^2\left[\sin^2\left(\frac{\alpha}{2}\right) + s^2\sin^2\left(\frac{\beta}{2}\right) - 2s\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right]$$

Go back to step 2 and repeat sequence until the difference between estimates of x is less than a desired threshold. In the performed calculations this threshold was set at 1×10^{-8}.

For values of s less than 1 ($s < 1$), Figure 12 applies where the labels a and b shown in Figure 11 are reversed.

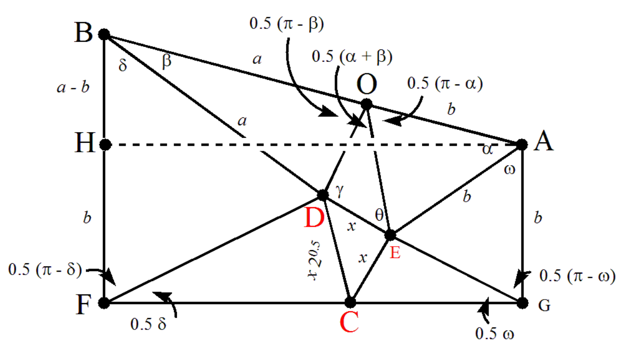

FIGURE 12. Diagram pertaining to the Morikawa problem for unequal-sized circles where $b < a$, or $s < 1$.

The steps in the iterative cycle change as shown by equations (30) to (38) starting with a set equal to 1 and varying s so that $s < 1$:

Step 1. Determine angle δ.

(30) $$\delta = \frac{\pi}{2} - \beta - \sin^{-1}\left(\frac{1-s}{s+1}\right)$$

Step 2. Determine length CF.

(31) $$\overline{CF} = a\sin\delta + \sqrt{a^2\left(\sin^2\delta - 4\sin^2\left(\frac{\delta}{2}\right)\right) + 2x^2}$$

Step 3. Determine angle DCF.

(32) $$\angle DCF = \sin^{-1}\left[\left(\frac{a\sqrt{2}}{x}\right)\sin^2\left(\frac{\delta}{2}\right)\right]$$

Step 4. Determine length GC.

(33) $$\overline{GC} = 2a\sqrt{s} - \overline{CF}$$

Step 5. Determine angle ECG.

(34) $$\angle ECG = \frac{3\pi}{4} - \angle DCF$$

Step 6. Determine length GE.

(35) $$\left(\overline{GE}\right)^2 = \left(\overline{GC}\right)^2 + x^2 - 2x\left(\overline{GC}\right)\cos\angle ECG$$

Step 7. Determine angle ω.

(36) $$\omega = 2\sin^{-1}\left(\frac{\overline{GE}}{2as}\right)$$

Step 8. Determine angle α.

(37) $$\alpha = \frac{\pi}{2} - \omega + \sin^{-1}\left(\frac{1-s}{s+1}\right)$$

Step 9. Obtain length ED = x (new estimate of x) from

(38) $$x^2 = 4a^2\left[s^2\sin^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\beta}{2}\right) - 2s\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right]$$

Go back to step 2 and repeat sequence until the difference between estimates of x is less than a desired threshold. In the performed calculations this threshold was set at 1 X 10^{-8}.

Tables S1 to S23 in the Appendix summarize the results of applying the iterative cycles for various starting values of s. In each case the minimum value of x was found by fitting an approximate quadratic function of the form of equation (20) for a range of β values in the vicinity of the sought after minimum.

Table 3 summarizes the values of the minimum x values found as a function of s. We note that when s = 1, we obtain the same minimum value of x(min) = 0.3855 found in Case I for a tilted square between to two equal-sized circles and a straight line (see Figure 9).

s	x(min)
1000000	1.411147
100000	1.405332
10000	1.387062
1000	1.329279
100	1.172895
50	1.091882
20	0.952919
10	0.835583
9	0.814244
8	0.795831
7	0.766498
6	0.735449
5	0.693113
4	0.655114
3	0.596574
2	0.516018
1	0.385505
0.5	0.258340
0.2	0.140037
0.1	0.083572
0.01	0.011699
0.001	0.001350
0.0001	0.000163

Figure 13 shows a plot of *x*(min) versus log *s* and the result of a non-linear least squares fitting of the data in Table 3 according to a sigmoid function of the same form as given by equation (12). The four coefficients are $a_1 = 0.5021 \pm 0.00418$, $a_2 = 0.3385 \pm 0.00500$, $a_3 = -0.0585 \pm 0.00266$, and $a_4 = 0.6410 \pm 0.00613$. At very large values of *s* the limiting value of *x*(min) approaches 0.5021/0.3385 = 1.4833.

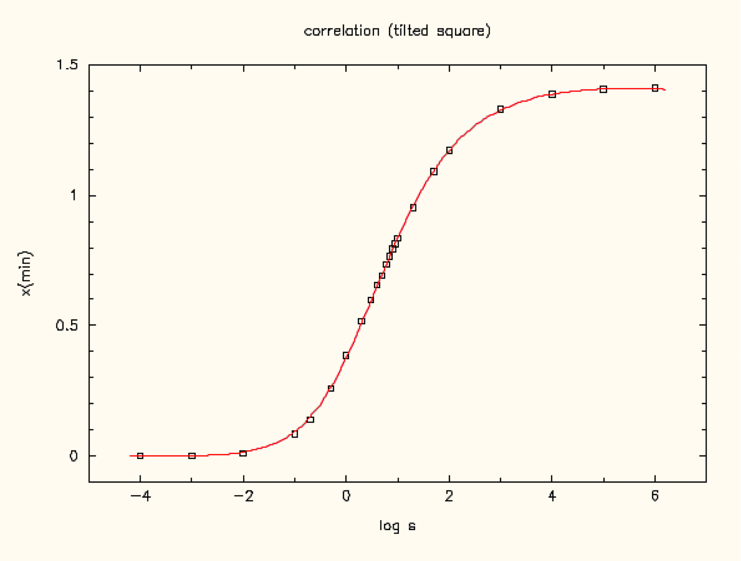

FIGURE 13. Data fit of *x*(min) values versus log *s* according to the sigmoidal expression given in equation (12) for a minimum tilted square between two circles and a tangent line.

On comparing the results shown in Tables 1 and 3, we observe that for every *s* value the length of the maximum horizontal oriented square is always larger than the length of the minimum tilted square. Table 4 summarizes data for the minimum length of the tilted square as a function of the ratio of radius lengths, *s*, as well as the values of angle β that result in minimum values of *x*.

Table 4. Summary of data for length of squares as a function of the ratio of radius lengths, *s*.

s	x(min)	(deg) angle beta	x(max)	difference
1000000	1.411147	0.000115	1.99583	0.584683
100000	1.405332	0.00114	1.986442	0.581110
10000	1.387062	0.0113	1.955432	0.568370
1000	1.329279	0.1095	1.8544	0.525112
100	1.172895	0.994	1.5629	0.390005
50	1.091882	1.89	1.4164	0.324518
20	0.952919	4.27	1.1835	0.230581
10	0.835583	7.47	0.9888	0.153217

9	0.814244	8.10	0.9588	0.144556
8	0.795831	8.79	0.9252	0.129369
7	0.766498	9.72	0.8872	0.120702
6	0.735449	10.94	0.8436	0.108151
5	0.693113	12.53	0.7925	0.099387
4	0.655114	14.58	0.7313	0.076186
3	0.596574	17.66	0.6548	0.058226
2	0.516018	22.56	0.5531	0.037082
1	0.385505	32.04	0.4	0.014495
0.5	0.258340	22.56	0.2765	0.018160
0.2	0.140037	12.50	0.1585	0.018463
0.1	0.083572	7.40	0.098884	0.015312
0.01	0.011699	0.993	0.015629	0.003930
0.001	0.001350	0.1094	0.001854	0.000504
0.0001	0.000163	0.0113	0.000196	3.3E-05

5. SUMMARY

On the basis of this investigation of the Morikawa sangaku problem we have the following conclusions:

1. Fitting a *horizontal* oriented square in the space between two unequal circles and a horizontal line that is tangent to both circles leads to a *maximum*-sized square.

2. Fitting a *tilted* square in the space between two unequal circles and a horizontal line that is tangent to both circles leads to a *minimum*-sized square under the constraint that the upper corners of the square touch the circles and the third corner touches the horizontal line.

3. For a given ratio of circle radii, the size of the tilted square is always smaller than the size of the horizontal square. The difference in size of the two kinds of squares increases as the ratio of the circle radii increases.

4. Iterative numerical computational results show that in both cases the dependence of x(max) or x(min) versus log b/a, where a and b are the radii of the circles follows a *sigmoidal* behaviour according to a form given by

$$x = \frac{a_1}{a_2 + 10^{-\left(a_3 z^2 + a_4 z\right)}} \text{, where } z = \log s = \log\left(b/a\right)$$

For very large $s = b/a$ values, upper plateaus of 2 and 1.4 are determined for x(max) and x(min), respectively.

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APPENDIX

Table S1. Summary of values of x for various tilted squares fit between two unequal circles with s = 0.0001.

(deg) angle β	x
0.01118	0.000138675940
0.01119	0.000138671854
0.0112	0.000138667855
0.01125	0.000138649257
0.0113	0.000138641782
0.01135	0.000138649795
0.0114	0.000138669244
0.01145	0.000138700429
0.0115	0.000138743658

Table S2. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 0.001.

(deg) angle β	x
0.1088	0.001329520
0.1089	0.001329494
0.109	0.001329476
0.1091	0.001329462
0.1092	0.001329452
0.1093	0.001329448
0.1094	0.001329448
0.1095	0.001329453
0.1096	0.001329462
0.1097	0.001329477
0.1098	0.001329496
0.1099	0.001329519

Table S3. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 0.01.

(deg) angle β	x
0.98	0.0117356
0.985	0.0117332
0.99	0.0117321
0.995	0.0117324
1	0.0117340
1.005	0.0117371
1.01	0.0117415

Table S4. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 0.1.

(deg) angle β	x
7.1	0.083890
7.2	0.083704
7.3	0.083596
7.4	0.083558
7.5	0.083590
7.6	0.083696
7.7	0.083875
7.8	0.084130
7.9	0.084460

Table S5. Summary of values of x for various tilted squares fit between two unequal circles with s = 0.2.

(deg) angle β	x
12	0.140582
12.2	0.140268
12.3	0.140170
12.4	0.140112
12.5	0.140094
12.6	0.140116
12.7	0.140179
12.8	0.140283
12.9	0.140428
13	0.140615

Table S6. Summary of values of x for various tilted squares fit between two unequal circles with s = 0.5.

(deg) angle β	x
21	0.260543
21.2	0.259962
21.4	0.259458
21.6	0.259032
21.8	0.258685
22	0.258416
22.2	0.258227
22.4	0.258118
22.6	0.258089
22.8	0.258141
23	0.258273
23.2	0.258487
23.4	0.258782
23.6	0.259159
23.8	0.259616

Table S7. Summary of values of x for various tilted squares fit between two unequal circles with s = 1.

(deg) angle β	x
29	0.391179
30	0.388028
31	0.386091
31.5	0.385589
32	0.385400

32.5	0.385527
33	0.385971
34	0.387809
35	0.390906

Table S8. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 2.

(deg) angle β	x
21	0.521086
21.5	0.518471
22	0.516832
22.25	0.516384
22.5	0.516186
22.75	0.516240
23	0.516547
23.5	0.517921
24	0.520309
24.5	0.523707
25	0.528105

Table S9. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 3.

(deg) angle β	x
16.5	0.602871
17	0.599061
17.5	0.597406
18	0.597775
18.5	0.600199
19	0.604687
19.5	0.611230
20	0.619801
20.5	0.630357
21	0.642839
21.5	0.657182

Table S10. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 4.

(deg) angle β	x
14	0.657923
14.5	0.655571
15	0.656586

15.5	0.661040
16	0.668959
16.5	0.680319
17	0.695057
17.5	0.713075
18	0.734246
18.5	0.758432

Table S11. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 5$.

(deg) angle β	x
12	0.702909
12.25	0.701070
12.5	0.700469
12.75	0.701131
13	0.703075
13.5	0.710861
14	0.723865
14.5	0.742034
15	0.765238
16	0.825977

Table S12. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 6$.

(deg) angle β	x
10.5	0.739611
11	0.736882
11.5	0.741193
12	0.752777
12.25	0.761323
12.5	0.771698
12.75	0.783886
13	0.797862
13.5	0.831061
14	0.871015
14.5	0.917431

Table S13. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 7$.

(deg) angle β	x
9.5	0.768437

10	0.768298
10.5	0.777661
11	0.796816
11.5	0.825784
12	0.864372
12.25	0.887177
12.5	0.912266
12.75	0.939598
13	0.969138
13.5	1.034754
14	1.109084

Table S14. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 8.

(deg) angle β	x
8.5	0.795447
9	0.793969
9.5	0.804596
10	0.827839
10.5	0.863802
11	0.912287
11.5	0.972971
12	1.045618
12.25	1.086435
12.5	1.130318
12.75	1.177379
13	1.227801

Table S15. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 9.

angle β	x
8	0.815754
8.5	0.821934
9	0.843683
9.5	0.881449
10	0.935197
10.5	1.004695
11	1.089905
11.5	1.191488
12	1.311644
12.25	1.380244

Table S16. Summary of values of *x* for various tilted squares fit between two unequal circles with *s* = 10.

(deg) angle β	x
7	0.846482
7.5	0.835905
8	0.848672
8.5	0.880786
9	0.932716
9.5	1.004468
10	1.096182
10.5	1.209008

Table S17. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 20$.

(deg) angle β	x
3.75	0.997602
4	0.967010
4.25	0.958543
4.5	0.971387
4.75	1.002751
5	1.054299
5.5	1.226335
6	1.543357

Table S18. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 50$.

(deg) angle β	x
1.85	1.093271
1.86	1.093028
1.875	1.092930
1.876	1.092935
1.88	1.092970
1.9	1.093496
2	1.105614
2.15	1.158071
2.25	1.220916

Table S19. Summary of values of *x* for various tilted squares fit between two unequal circles with $s = 100$.

(deg) angle β	x
0.975	1.174281

1	1.173405
1.01	1.174153
1.02	1.175482
1.03	1.177413
1.04	1.179970
1.05	1.183176
1.1	1.209911

Table S20. Summary of values of x for various tilted squares fit between two unequal circles with s = 1000.

(deg) angle β	x
0.1085	1.329820
0.109	1.329476
0.1095	1.329453
0.11	1.329548
0.111	1.330105
0.112	1.331170
0.113	1.332769
0.114	1.334929
0.115	1.337679

Table S21. Summary of values of x for various tilted squares fit between two unequal circles with s = 10000.

(deg) angle β	x
0.0112	1.386679
0.01125	1.386493
0.011275	1.386420
0.0113	1.386418
0.01135	1.386498
0.0114	1.386692
0.0115	1.387437
0.0116	1.388675
0.0118	1.392749
0.012	1.399160

Table S22. Summary of values of x for various tilted squares fit between two unequal circles with s = 100000.

(deg) angle β	x
0.00114	1.405319
0.001141	1.405319
0.001142	1.405323

0.001143	1.405332
0.001144	1.405345
0.001146	1.405385
0.001148	1.405442

Table S23. Summary of values of *x* for various tilted squares fit between two unequal circles with s = 1000000.

(deg) angle β	x
0.00011430	1.4113939710
0.00011435	1.4113918200
0.00011440	1.4113904290
0.00011444	1.4113902360
0.00011445	1.4113902770
0.00011450	1.4113915570
0.00011500	1.4114635170
0.00011600	1.4119447400
0.00011700	1.4128959540
0.00011800	1.4143435100
0.00012000	1.4188448470